

Multi-black rings and the phase diagram of higher-dimensional black holes

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Abstract

Configurations of multiple concentric black rings play an important role in determining the pattern of branchings, connections and mergers between different phases of higher-dimensional black holes. We examine them using both approximate and (in five dimensions) exact methods. By identifying the role of the different scales in the system, we argue that it is possible to have multiple black ring configurations in which all the rings have equal temperature and angular velocity. This allows us to correct and improve in a simple, natural manner, an earlier proposal for the phase diagram of singly-rotating black holes in $D \geq 6$.

1 Introduction

As a result of recent progress, a complete understanding of the possible phases of some classes of higher-dimensional black holes appears now as a realistic objective (some, but not all, of this progress, is reviewed in [1]). Here we focus on the most basic sector of vacuum, asymptotically flat black holes. In five dimensions, there appears to be little room for new solutions besides the basic ones of the Myers-Perry black holes [2], black rings [3, 4], and helical black rings [5], and combinations of all these [6, 7, 8, 9, 10].

The situation in $D \geq 6$, although much less developed than in five dimensions, has also seen significant advances. In this paper we focus on the sector of configurations with rotation in a single plane. This is simple enough that we may conceivably narrow down the spectrum of possibilities until only some details of branches and connections remain to be determined, possibly through numerical work¹. In this direction, ref. [11] identified a large portion of the possible phases and advanced proposals for the connections among them.

An important ingredient in the phase diagram are configurations involving several concentric black rings. These are particularly relevant for understanding branchings and connections among phases. It was observed in [12] that at sufficiently large rotation, the class of Myers-Perry black holes should branch off into new phases where the horizon develops axisymmetric ‘pinches’ (this has been subsequently confirmed in [13]). As one moves along one of the new branches of pinched black hole solutions, it is expected that these pinches grow and eventually pinch down to a singularity on the horizon. Continuing the evolution in phase space, the horizon develops into a non-simply connected horizon or splits into disconnected components. The simplest instance of this phenomenon is a black hole developing a pinch along its axis of rotation and connecting to a black ring. The next simplest case is a black hole with a circular axisymmetric pinch giving rise to a black Saturn. This picture was developed in [11] by combining evidence from a variety of strands.

In the evolution in phase space of a configuration with a connected horizon (I) that develops into a configuration (II) with two disconnected black holes through an intermediate singular solution (I’), it is important to realize that, since these are equilibrium configurations, the temperature T (*i.e.*, surface gravity) and angular velocity Ω_H of the horizon must remain uniform on the horizon of phase (I) and by continuity in phase (I’) too². It is then natural to expect that the evolution can continue into a configuration (II) where the two separate horizons still have equal temperatures and angular velocities. This is, the branch of pinched black holes continues into a branch of multiple black holes and black rings which are in *thermodynamic equilibrium*.

There is nothing wrong with a classical stationary configuration of multiple black holes

¹For the purposes of this paper, a ‘branch of solutions’ means a one-parameter family of black holes (possibly with multiple, disconnected horizons) with fixed total mass and varying total angular momentum. It is adequately represented as a curve of area vs. angular momentum, at fixed mass.

²Note that we are not demanding that T and Ω_H remain constant as we move from one solution to another in the same branch, but rather that they are uniform over the horizon of each solution.

where each horizon has a different surface gravity and angular velocity. However, if quantum effects are turned on, then Hawking radiation will put these black holes in contact with each other and the radiation will tend to equilibrate both their temperatures and their angular velocities. Of course, here we are glossing over the fact that the Hartle-Hawking state of thermodynamic equilibrium typically does not exist for rotating black holes due to the presence of ergoregions [14]. But this is not an issue for our motivation to impose that the surface gravities and angular velocities remain uniform in multi-horizon configurations across the connection between two phases. This has nothing to do with actual temperature and radiation, but, given the obvious thermodynamic analogies, we will continue to refer to the configurations of our interest as being in thermodynamic equilibrium.

Rotating black holes that develop several concentric pinches should naturally evolve into configurations with several concentric black rings in thermodynamic equilibrium (possibly with a central black hole of spherical topology). However, ref. [11] found an apparent objection against the existence of multi-rings in thermodynamic equilibrium, which made it difficult to complete the phase diagram. The main purpose of this paper is to revisit the properties of multi-ring configurations, identify the loophole in the arguments in [11], and then propose a new, simpler form for the phase diagram of singly-rotating black holes in $D \geq 6$. The arguments can be put to the test by studying the five-dimensional exact di-ring solutions constructed in [7] (see also [8]). Even if the details of the connections differ in five dimensions (where all the mergers occur on a horizonless singular solution) and in six or more dimensions, the properties of multi-rings at very large angular momentum remain qualitatively similar.

In the next section we study multi-black rings in the approximation where they are thin. Such methods, more generally developed in [15], had already been employed in an analysis of five-dimensional black Saturns in [16], and give us the necessary understanding of the physics of these configurations. In section 3 we check the validity of these methods against the exact five-dimensional black di-rings. We also obtain the properties of di-rings at all angular momenta, which is not possible within the thin-ring approximation. Section 4 gathers together the new information to draw what we believe is a qualitatively correct phase diagram. We close in Section 5 with some additional remarks. The appendix contains details of the properties of the exact five-dimensional di-ring.

Note: when this project was in its final stages we learned from T. Mishima of his work with H. Iguchi that overlaps with section 3.

2 Multi-black rings in thermodynamic equilibrium

2.1 Thin-ring approximation

We say that a black ring in D dimensions, with horizon topology $S^1 \times S^{D-3}$, is thin when the S^1 radius R is much longer than the S^{D-3} radius r_0 that measures the ring ‘thickness’. In this section we will work only to leading order in r_0/R , in which case these two radii can

be unambiguously defined. The physical properties of black rings in this limit have been determined in [11] as

$$\begin{aligned} M &\rightarrow \frac{\Omega_{(D-3)}(D-2)}{8} R r_0^{D-4}, & J &\rightarrow \frac{\Omega_{(D-3)}\sqrt{D-3}}{8} R^2 r_0^{D-4}, \\ A_H &\rightarrow 2\pi\Omega_{(D-3)}\sqrt{\frac{D-3}{D-4}} R r_0^{D-3}, \end{aligned} \quad (2.1)$$

$$T \rightarrow \frac{D-4}{4\pi} \sqrt{\frac{D-4}{D-3}} \frac{1}{r_0}, \quad \Omega_H \rightarrow \frac{1}{\sqrt{D-3}} \frac{1}{R}. \quad (2.2)$$

(we set $G = 1$). We introduce a dimensionless angular momentum and area appropriate for comparing different black objects with a given mass,

$$j = c_J \frac{J}{M^{\frac{D-2}{D-3}}}, \quad a_H = c_A \frac{A_H}{M^{\frac{D-2}{D-3}}} \quad (2.3)$$

where the D -dependent numerical constants c_J and c_A have been chosen in [11, 1] as

$$c_J = \frac{\Omega_{D-3}}{2^{D+1}} \frac{(D-2)^{D-2}}{(D-3)^{\frac{D-3}{2}}}, \quad c_A = \frac{\Omega_{D-3}}{2(16\pi)^{D-3}} (D-2)^{D-2} \left(\frac{D-4}{D-3}\right)^{\frac{D-3}{2}} \quad (2.4)$$

but these specific values are actually irrelevant to our arguments. Thus, for a single thin black ring

$$j \rightarrow 2^{-\frac{D-2}{D-3}} \left(\frac{R}{r_0}\right)^{(D-4)/(D-3)}, \quad (2.5)$$

and

$$a_H^{(\text{single-ring})} \rightarrow 2^{\frac{D-6}{(D-4)(D-3)}} j^{-1/(D-4)}. \quad (2.6)$$

The thin ring limit is then equivalent to a large j limit. The separation of the length scales $r_0 \ll R$ implies that the thin ring can be regarded as a black string curved on a circle of large radius. To leading order, the gravitational interaction among segments of the string separated by a distance $\sim R$ can be neglected.

2.2 Multiple thin rings and thermal equilibrium

The quantities a_H and j can also be introduced for generic multi-black hole phases, where M , J and A_H are the total mass, spin and area of the entire system. As argued in [16], for a system with n black objects (*i.e.*, concentric black rings with or without a central black hole) specifying the total M and J leaves $2n - 2$ continuous parameters undetermined. In general these are configurations where the temperature and angular velocities of the multiple horizons, $T^{(i)}$, $\Omega_H^{(i)}$ are different and so the system is out of thermodynamic equilibrium, which is perfectly acceptable for a stationary classical black hole system. However, as discussed in the introduction we will be particularly interested in thermodynamic equilibrium where

$$T^{(i)} = T^{(j)}, \quad \Omega_H^{(i)} = \Omega_H^{(j)} \quad \forall i, j. \quad (2.7)$$

These are $2n - 2$ conditions, which remove all the continuous non-uniqueness. Thus multi-black hole phases in thermal equilibrium correspond to curves $a_H(j)$.

The thin ring approximation can be used in order to describe black Saturns in which the radius of the central black hole r_{bh} is much smaller than the ring's S^1 radius R , so the interaction between the two objects is negligible and the total mass, spin, and area are just the sum of the two separate components [16]. It is easy to deduce that for a thermally-equilibrated black Saturn in any number of dimensions, the dimensionless total area $a_H(j)$ asymptotes (from below) to the same value as for a single black ring

$$\frac{a_H^{(\text{Saturn})}}{a_H^{(\text{single-ring})}} \xrightarrow{j \rightarrow \infty} 1. \quad (2.8)$$

The reason is that equality of the temperatures forces the black hole radius r_{bh} to be of the same order as the ring's r_0 , but the mass, spin and area of the ring are much larger since they contain factors of $R \gg r_0$ so the black hole contributes negligibly to them.

We now extend these arguments to understand the properties of multi-black ring systems. To begin with, let us consider two thin black rings with radii $R^{(2)} < R^{(1)}$ and horizon thicknesses $r_0^{(1)}, r_0^{(2)}$. The interaction energy between them will be smaller than their masses as long as

$$r_0^{(1,2)} \ll R^{(1)} - R^{(2)}. \quad (2.9)$$

Then we can approximate, to leading order in $r_0^{(i)}/R^{(i)}$ and in $r_0^{(i)}/(R^{(1)} - R^{(2)})$, the total mass and angular momentum of the system as

$$M \simeq M^{(1)} + M^{(2)}, \quad J \simeq J^{(1)} + J^{(2)}. \quad (2.10)$$

The total area is always the sum of individual areas, $A_H = A_H^{(1)} + A_H^{(2)}$. Fixing M and J leaves two continuous parameters unspecified in the di-ring configuration, which could be *e.g.*, the mass and spin of one of the black rings.

If we impose the two conditions that the temperatures and angular velocities of the two black rings are equal, then the doubly-continuous degeneracy is removed. However, since fixing T fixes the thickness r_0 and fixing Ω_H fixes the radius R (see (2.2)), naively it would appear that thermal equilibrium on thin black di-rings requires that the two rings should actually be the same, *i.e.*, they would lie on top of each other. In other words, the interaction between them would be very strong so the initial approximation breaks down, and conceivably thermally-equilibrated di-rings might not be allowed [11].

What this argument misses is that it is possible to have a hierarchy between the three main scales in the problem,

$$r_0 \ll R^{(1)} - R^{(2)} \ll R^{(i)} \quad (2.11)$$

such that one can consistently neglect the interaction between the two rings while at the same time satisfying the thermal equilibrium conditions³.

³Here $R^{(1)}$ and $R^{(2)}$ are of the same parametric order, while $r_0^{(1)}$ and $r_0^{(2)}$ are equal to leading order of approximation.

The way to conceive of this configuration is by first considering two parallel boosted black strings, with equal horizon thickness r_0 and equal boost parameters. Then, the two black strings have the same temperature and horizon velocity, and if the distance between them is $L \gg r_0$ then we can linearly superimpose the solutions to a good approximation (precisely, $O(r_0/L)^{D-3}$). Next, we bend this di-string into a large circle of radius $R \gg L$ so the interaction of segments of the di-string at distance $\sim R$ can be neglected. The balance between the tension of the di-ring and the centrifugal force can easily be seen to fix the boost value to the same parameter as in a single ring, and we can obtain the physical properties as in (2.10). Clearly, since $L = R^{(1)} - R^{(2)}$ this is the regime (2.11). In the language of the blackfold effective-theory approach, we integrate successively the physics at two scales, first r_0 (the thickness of the black strings), then $R^{(1)} - R^{(2)}$ (the ‘thickness’ of the di-ring).

Under these conditions, the mass, spin and area of the two black rings (which in our approximation are well-defined quantities) are equal to leading order in the approximation (2.11),

$$M^{(1)} \simeq M^{(2)}, \quad J^{(1)} \simeq J^{(2)}, \quad A_H^{(1)} \simeq A_H^{(2)}, \quad (2.12)$$

so for the entire di-ring system we have

$$j = c_J \frac{2J^{(1)}}{(2M^{(1)})^{\frac{D-2}{D-3}}} = 2^{-1/(D-3)} j^{(1)}, \quad (2.13)$$

$$a_H = c_A \frac{2A_H^{(1)}}{(2M^{(1)})^{\frac{D-2}{D-3}}} = 2^{-1/(D-3)} a_H^{(1)} \quad (2.14)$$

where $j^{(1)}$ and $a_H^{(1)}$ are the values for each individual ring. Since from (2.6) we know how $a_H^{(1)}$ depends on $j^{(1)}$, we can easily determine the asymptotic form of $a_H(j)$ for a thermally-equilibrated di-ring at large total angular momentum j . The result is that, for a single black ring and a black di-ring with the same total mass and spin, the dimensionless areas asymptote to a finite, non-zero ratio,

$$\frac{a_H^{(\text{di-ring})}}{a_H^{(\text{single-ring})}} \xrightarrow{j \rightarrow \infty} 2^{-1/(D-4)}. \quad (2.15)$$

The argument can easily be extended to n black ring configurations in thermal equilibrium when they satisfy the condition that

$$r_0 \ll R^{(i)} - R^{(j)} \ll R^{(i)} \quad (2.16)$$

for all pairs $R^{(i)} > R^{(j)}$ so their mutual interactions are negligible. Then

$$\frac{a_H^{(n\text{-ring})}}{a_H^{(\text{single-ring})}} \xrightarrow{j \rightarrow \infty} n^{-1/(D-4)}. \quad (2.17)$$

Finally, it is equally straightforward to argue that for a black Saturn with n black rings in thermodynamic equilibrium, a_H is asymptotically the same at large j as for the n -ring without a central black hole. This is, the asymptotic behavior of $a_H(j)$ only depends on the number of rings in the system.

3 Exact results from di-rings in five dimensions

In five dimensions we are not constrained to the approximation where the interaction between the black rings is small. Exact di-ring solutions have been constructed in [7, 8] using integrability techniques. In the appendix we give the exact expressions for the total M , J , A_H of the configuration, as well as the individual $A_H^{(i)}$, $T^{(i)}$, $\Omega_H^{(i)}$, which are well-defined magnitudes on each horizon. In five dimensions, the dimensionless magnitudes that we use to describe the phase space at fixed mass are conventionally normalized as

$$a_H = \frac{3}{16} \sqrt{\frac{3}{\pi}} \frac{A_H}{M^{3/2}}, \quad j = \sqrt{\frac{27\pi}{32}} \frac{J}{M^{3/2}}. \quad (3.1)$$

As mentioned above, fixing M and J leaves two parameters, *e.g.*, the mass and spin of one of the black rings, unspecified. We can easily argue that this allows di-rings to take any value of j and a_H within the band

$$-\infty < j < \infty, \quad 0 < a_H < 1. \quad (3.2)$$

The upper bound $a_H = 1$ is set by the maximum area that a single black ring can have. The argument is similar to the one employed in [16] for black Saturns: we can consider a central ring with area arbitrarily close to $a_H = 1$, and then surround it with a long and thin ring that carries arbitrarily little entropy and mass but which can contribute an arbitrary large amount of angular momentum to the total configuration by making its radius as long as required. The parameters can be adjusted to reproduce any value of j and a_H within the band (3.2). Indeed, typically there is a one-parameter family of di-rings at any given point in (3.2).

As before, the conditions of thermodynamical equilibrium

$$T^{(1)} = T^{(2)}, \quad \Omega_H^{(1)} = \Omega_H^{(2)} \quad (3.3)$$

remove completely the continuous non-uniqueness (leaving at most discrete degeneracies) so these phases are characterized by curves $a_H(j)$. The resulting phase diagram is depicted in fig. 1.

The most notable feature of this diagram is that the different curves all join at the singular extremal black ring at $j = 1$, $a_H = 0$, and that the area curve of black di-rings always lies below the corresponding black ring curve. Let us focus on the large- j asymptotic behavior, as obtained from the explicit exact solutions. For a single black ring the asymptotic value is (see *e.g.*, [17])

$$a_H^{(\text{single-ring})} = \frac{1}{\sqrt{2}j} + O(j^{-3}), \quad (3.4)$$

which is known to be reproduced exactly by the thin-ring (blackfold) methods of [11, 15].

Using the exact results in the appendix, we can obtain the asymptotic form of the di-ring curve in the limit of large j . We find⁴

$$a_H^{(\text{di-ring})} = \frac{1}{2\sqrt{2}j} + O(j^{-7/3}). \quad (3.5)$$

⁴The first subleading terms are $a_H = \frac{1}{2\sqrt{2}j} + \frac{3}{32 \times 2^{5/6}} \frac{1}{j^{7/3}} + \frac{1}{16\sqrt{2}} \frac{1}{j^3} + O(j^{-11/3})$.

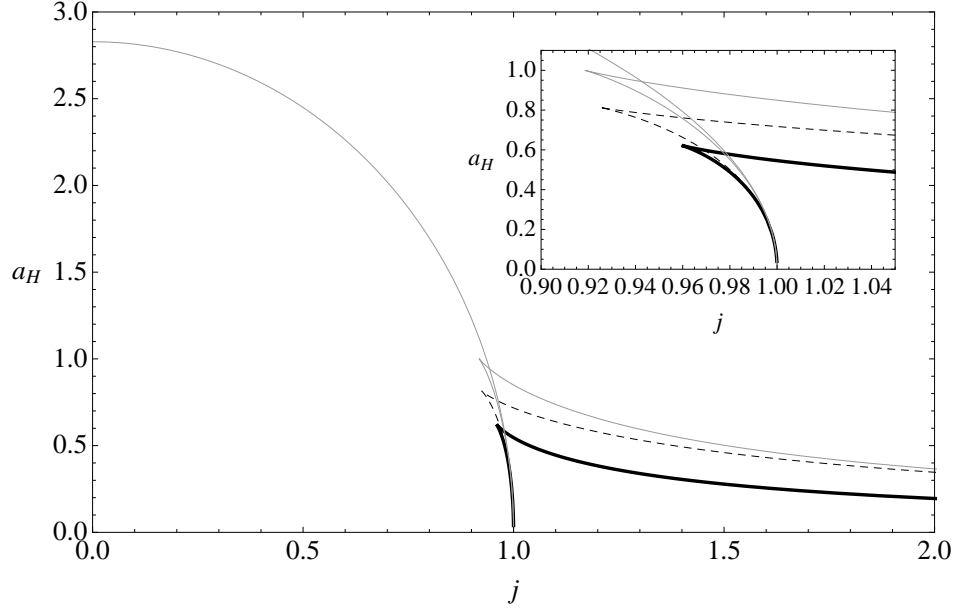


Figure 1: Singly-rotating phases in thermodynamical equilibrium in five dimensions. The gray curves correspond to the singly-spinning Myers-Perry black hole and the black ring, the dashed curve corresponds to the black Saturn and the black curve to the di-ring. For the black Saturn, the maximum of the total area and minimum of the spin is at $(a_H^{(\text{Saturn})}, j) \approx (0.814, 0.9245)$ [6]. For the di-ring it is at $(a_H^{(\text{di-ring})}, j) \approx (0.623, 0.9596)$. The large- j asymptotic behavior predicted in (2.8), (2.15) can be discerned already at $j \sim 2$.

We see that our general argument that yielded the asymptotic result (2.15) indeed reproduces the correct value in $D = 5$.

We can further check that in this configuration the separation of scales (2.11) occurs as $j \rightarrow \infty$. Using the exact solution of [8], we define⁵

$$R^{(1),(2)} = \sqrt{g_{\psi\psi}} \Big|_{\rho=0, z=a_{2,5}}, \quad r_0 = \sqrt{g_{\phi\phi}} \Big|_{\rho=0, z=\frac{a_2+a_3}{2}}. \quad (3.6)$$

In the limit of large j the difference between alternative definitions of these quantities (see [18]) vanishes as a positive power of $1/j$. From the exact solution we obtain

$$\begin{aligned} \frac{r_0}{R^{(1)} - R^{(2)}} &= \frac{1}{8 \times 2^{5/6}} \frac{1}{j^{4/3}} - \frac{1}{64\sqrt{2}} \frac{1}{j^2} + O(j^{-8/3}), \\ \frac{R^{(1)} - R^{(2)}}{R^{(1)}} &= \frac{1}{2^{2/3}} \frac{1}{j^{2/3}} - \frac{1}{4 \times 2^{1/3}} \frac{1}{j^{4/3}} + \frac{3}{16j^2} + O(j^{-8/3}), \end{aligned} \quad (3.7)$$

so we are indeed in the regime (2.11). Thus our assumption that at large j the di-ring exists in a thermodynamic equilibrium configuration where the interactions between rings are weak, is well-supported by the exact solution.

⁵Thus we take R as the radius of the S^1 at the outer rim of the black ring horizon, and r_0 as the radius of the S^2 at the (coordinate) mid-point between the poles of the horizon S^2 of the outer black ring.

4 Phase diagram revisited

It was conjectured in [12], and recently confirmed in [13], that for a sufficiently large spin, singly-rotating black holes in $D \geq 6$ branch off into new phases of stationary black holes with axisymmetric ‘pinched’ horizons.

It is expected that moving in the space of solutions along a new branch of stationary black holes, the deformation increases until the horizon pinches down to zero size at the axis of rotation or at a circle centered on the axis. The simplest instance corresponds to the merger of black holes with a single pinch along the rotation axis, with ‘fat’ black rings whose central hole has shrunk to zero size. Similarly, black holes with one circular pinch are expected to merge with black Saturns. An important point here is that, if we follow the evolution in phase space of these solutions, then coming from the side of the pinched black hole the temperature and angular velocity must be uniform on the horizon. By continuity, this will also be the case at the merger point and then it is natural to expect that the phases branch off into configurations of black Saturns in thermal equilibrium. That is, thermal equilibrium phases of multi-black holes are important for determining the phase structure near merger points.

Using this information, ref. [11] made a proposal for the phase diagram of singly-spinning black objects in $D \geq 6$. Some of the qualitative structure of the conjectured diagram is robust—these are the solid lines and figures in fig. 6 of ref. [11]. However, there were some phases and connections that remained more uncertain—these are the dashed lines and figures in fig. 6 of ref. [11]. They were motivated by the apparent absence of multi-rings in thermal equilibrium, which we can now assert is incorrect.

A first amendment to the picture in [11] has been already made in ref. [5]. Using the blackfold approach it has been argued that the ‘pancaked black Saturns’ conjectured in [11] cannot exist in thermal equilibrium: an ultraspinning black hole has larger radius than a black ring with the same temperature and angular velocity, and therefore would engulf it. Fortunately, our analysis in this paper shows that these pancaked Saturns are not actually needed to complete the phase diagram and describe the mergers of black holes with several pinches. Instead these are naturally connected to multi-black rings in thermal equilibrium. The corrected phase diagram is presented in fig. 2.

It must be noted that we cannot decide at present whether two pinches in a black hole will shrink to zero size simultaneously as one moves along the curve $a_H(j)$ in the phase diagram. It may well happen that one of the pinches reaches zero size before the other, thus giving rise to a phase where a pinched black hole is surrounded by a black ring, or instead one where a pinched black ring surrounds a central black hole. Ascertaining this probably requires explicit numerical construction of the solutions. Up to these uncertainties about the details of how the different phases actually connect, we believe that the qualitative features presented in fig. 2 are robust.

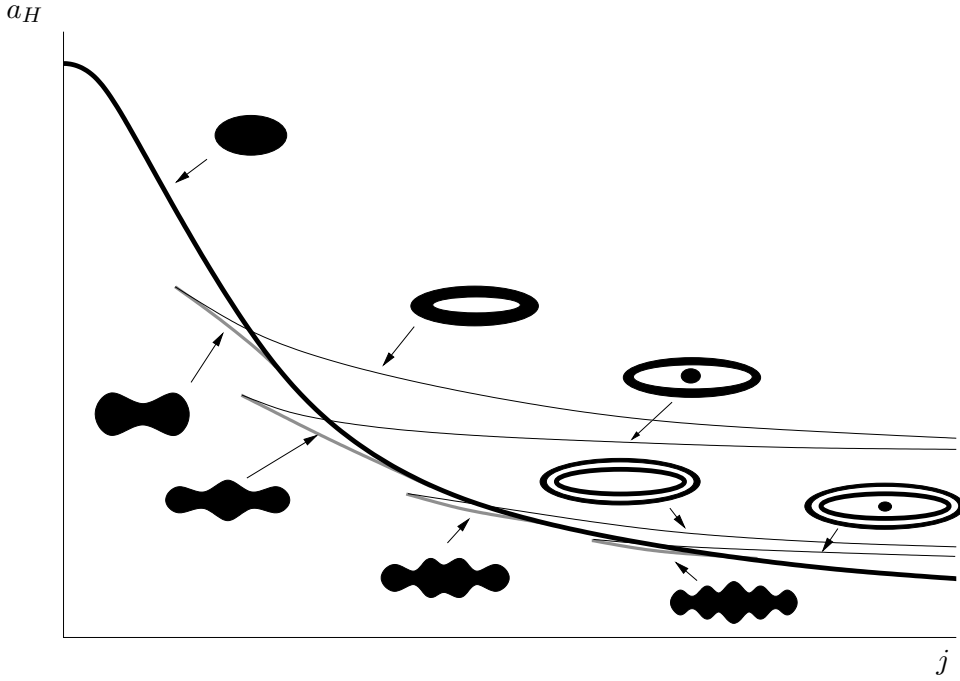


Figure 2: New proposal for the phase diagram of thermal equilibrium phases in $D \geq 6$. As in ref. [11], the details of the phase connections are unknown and smooth connections (*i.e.*, second order transitions) are possible instead of swallowtails with cusps (*i.e.*, first order transitions). In phases of black holes with multiple pinches evolving into multi-black rings (and multi-ring Saturns) it is also unknown whether intermediate pinched Saturns or pinched multi-rings appear (this depends on how the different pinches evolve along the phase curve). Other than this, the features in the diagram are robust. The asymptotic behavior of the curves depends only on the total number of rings and is given by eq. (2.17).

5 Final remarks

An incorrect assumption about the properties of multi-black rings in thermal equilibrium had led ref. [11] to conjecture an unduly complicated phase diagram in $D \geq 6$. In this paper, after properly identifying the possible hierarchies between the length scales in the system, we have concluded that the simplest and most natural completion of the diagram can actually be realized. An analysis of the exact five-dimensional di-ring solutions confirms in detail the results obtained by performing a thin-ring approximation.

While in $D \geq 6$ we have worked only to leading order in the thin-ring approximation, it should not be too difficult to estimate the size of subleading corrections in $r_0/(R^{(1)} - R^{(2)})$ by considering a linearized (Newtonian) approximation to the gravitational interaction between black objects.

We have only studied configurations with a single angular momentum. But arguments similar to the ones in this paper can be made for other multi-black hole systems, where instead of singly-spinning black rings we have doubly-spinning black rings [4], helical black rings, or blackfolds with horizon topology $\prod_{i|p_i \in \text{odd}} S^{p_i} \times S^{D-\sum_i p_i-2}$ [5].

Observe that the five-dimensional black Saturn and di-ring phases merge with other black hole phases (MP black hole, black ring) at the singular extremal solution at $j = 1$, $a_H = 0$. Presumably, this is also the case for all multi-ring phases, and it suggests that bifurcations of horizons (at least singly-rotating ones) do not occur in five dimensions. Indeed, it has been proven that singly-rotating MP black holes in $D = 5$ do not bifurcate [19], and this is likely the case for black rings too.

Acknowledgments

RE thanks Takashi Mishima for conversations and communications about their work overlapping with ours. PF thanks Oscar J.C. Dias, Ricardo Monteiro and Jorge E. Santos for many discussions about di-rings in thermodynamic equilibrium. RE thanks Yukawa Institute, Kyoto University (during the long-term workshop “Gravity and Cosmology 2010”) and KIAS (Seoul) for hospitality during the last phases of this work. RE is partially supported by DURSI 2009 SGR 168, MEC FPA 2007-66665-C02 and CPAN CSD2007-00042 Consolider-Ingenio 2010. PF is supported by an STFC rolling grant.

A The black di-ring

In this appendix we collect some of the physical parameters of the black di-ring that are needed in our analysis. The solution was first presented in [7] but we follow mostly the inverse-scattering construction of [8], to which we refer for more details. The rod structure is as in fig. 3, and the solution is parametrized by seven real numbers a_i with dimensions of length² that correspond to the rod endpoints and satisfy

$$a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < a_7. \quad (\text{A.1})$$

In this parametrization the black ring horizons lie at $\rho = 0$ and $z \in [a_2, a_3]$ and $z \in [a_5, a_6]$ respectively, where ρ and z are the standard Weyl coordinates.

The ADM mass and angular momentum of the solution are [8]:

$$\begin{aligned} M &= \frac{3\pi}{4} (a_3 - a_1 + a_6 - a_4), \\ J &= \frac{\pi}{2} \frac{(a_2 - a_1)(a_5 - a_1)c_1 + (a_4 - a_2)(a_5 - a_4)c_2}{a_4 - a_1}, \end{aligned} \quad (\text{A.2})$$

where c_1 and c_2 are constants which are fixed by demanding regularity of the solution:

$$c_1 = \sqrt{\frac{2(a_3 - a_1)(a_6 - a_1)(a_7 - a_1)}{(a_2 - a_1)(a_5 - a_1)}}, \quad c_2 = \sqrt{\frac{2(a_4 - a_3)(a_6 - a_4)(a_7 - a_4)}{(a_4 - a_2)(a_5 - a_4)}}. \quad (\text{A.3})$$

Once c_1 and c_2 have been fixed as in (A.3), the absence of conical singularities imposes two further constraints on the parameters of the solution [8]:

$$\frac{|Y - c_1 c_2 Z|}{\sqrt{X}} = 1, \quad \frac{(a_7 - a_1)(a_7 - a_4)(a_7 - a_3)(a_7 - a_6)}{(a_7 - a_2)^2(a_7 - a_5)^2} = 1, \quad (\text{A.4})$$

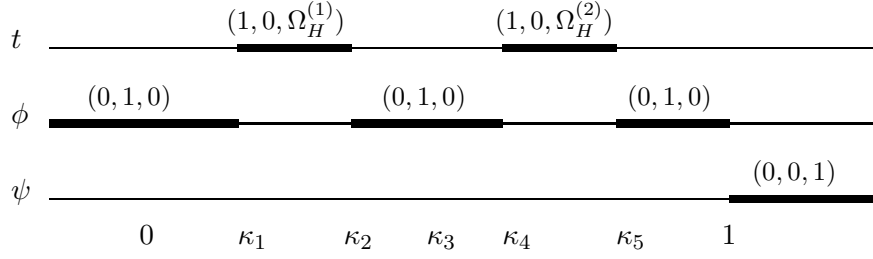


Figure 3: Rod structure of the di-ring.

where

$$\begin{aligned}
X &= \frac{4(a_4 - a_1)^2(a_5 - a_2)^2(a_6 - a_3)^2(a_7 - a_2)^2(a_4 - a_3)(a_6 - a_1)(a_7 - a_1)}{(a_4 - a_2)(a_5 - a_1)(a_5 - a_3)(a_6 - a_2)(a_7 - a_3)}, \\
Y &= 2(a_4 - a_3)(a_6 - a_1)(a_7 - a_1), \\
Z &= (a_2 - a_1)(a_5 - a_4).
\end{aligned} \tag{A.5}$$

The conditions in (A.4) imply that the general (regular) di-ring is described by an overall length scale and three dimensionless parameters [8]. As discussed in the main text, this number of parameters is to be expected since a black ring in equilibrium is uniquely specified by two (dimensionful) parameters, namely the temperature T and the angular velocity Ω_H . In the following we will consider only di-rings for which (A.3) and (A.4) have been imposed.

Following [6] (see also [8]) a better parametrization of the solution can be obtained by introducing an overall length scale L ,

$$L^2 = a_7 - a_1, \tag{A.6}$$

and five dimensionless constants that specify the relative lengths of the rods:

$$\kappa_i = \frac{a_{i+1} - a_1}{L^2}, \quad i = 1, \dots, 5, \tag{A.7}$$

so that $0 < \kappa_1 < \kappa_2 < \kappa_3 < \kappa_4 < \kappa_5 < 1$. In fig. 3 we depict the rod structure in terms of this parametrization. In addition, we find it useful to define

$$\bar{c}_1 = \frac{c_1}{L} = \sqrt{\frac{2\kappa_2\kappa_5}{\kappa_1\kappa_4}}, \quad \bar{c}_2 = \frac{c_2}{L} = \sqrt{\frac{2(1-\kappa_3)(\kappa_3-\kappa_2)(\kappa_5-\kappa_3)}{(\kappa_3-\kappa_1)(\kappa_4-\kappa_3)}}. \tag{A.8}$$

For the purpose of studying di-rings in thermodynamical equilibrium we will need some of the physical parameters of the individual black rings. We label the outer ring by “1” and the inner one by “2”. The physical parameters that we need are:

Horizon areas

$$\begin{aligned}
A_H^{(1)} &= L^3 4\sqrt{2}\pi^2 \sqrt{\frac{(\kappa_2 - \kappa_1)^3(\kappa_5 - \kappa_1)}{\kappa_3^2(1 - \kappa_1)^2(\kappa_4 - \kappa_1)^2}} \\
&\quad \times \sqrt{\kappa_1(\kappa_3 - \kappa_2)(1 - \kappa_3)(\kappa_4 - \kappa_3)(\kappa_5 - \kappa_3) + \kappa_2\kappa_4\kappa_5(\kappa_3 - \kappa_1) - \bar{c}_1\bar{c}_2\kappa_1\kappa_4(\kappa_3 - \kappa_1)(\kappa_4 - \kappa_3)},
\end{aligned} \tag{A.9a}$$

$$A_H^{(2)} = L^3 4\sqrt{2}\pi^2 \sqrt{\frac{\kappa_5(1 - \kappa_2)(1 - \kappa_3)(\kappa_5 - \kappa_1)(\kappa_5 - \kappa_3)(\kappa_5 - \kappa_4)^3}{(1 - \kappa_1)^2(1 - \kappa_4)^2(\kappa_5 - \kappa_2)^2}}, \tag{A.9b}$$

and the total horizon area is $A_H = A_H^{(1)} + A_H^{(2)}$.

Temperatures

$$T^{(1)} = \frac{1}{2\pi L} \sqrt{\frac{\kappa_3^2(1-\kappa_1)^2(\kappa_4-\kappa_1)^2}{2(\kappa_2-\kappa_1)(\kappa_5-\kappa_1)}} \times [\kappa_1(\kappa_3-\kappa_2)(1-\kappa_3)(\kappa_4-\kappa_3)(\kappa_5-\kappa_3) + \kappa_2\kappa_4\kappa_5(\kappa_3-\kappa_1) - \bar{c}_1\bar{c}_2\kappa_1\kappa_4(\kappa_3-\kappa_1)(\kappa_4-\kappa_3)]^{-1/2} \quad (\text{A.10a})$$

$$T^{(2)} = \frac{1}{2\pi L} \sqrt{\frac{(1-\kappa_1)^2(1-\kappa_4)^2(\kappa_5-\kappa_2)^2}{2\kappa_5(1-\kappa_2)(1-\kappa_3)(\kappa_5-\kappa_1)(\kappa_5-\kappa_3)(\kappa_5-\kappa_4)}} \quad (\text{A.10b})$$

Angular velocities

$$\Omega_H^{(1)} = \frac{1}{L} \frac{\bar{c}_1 \kappa_1 \kappa_3}{2 \kappa_2 \kappa_5 - \bar{c}_1 \bar{c}_2 \kappa_1 (\kappa_4 - \kappa_3)}, \quad (\text{A.11a})$$

$$\Omega_H^{(2)} = \frac{1}{2L} \frac{\bar{c}_1 \kappa_1 \kappa_4 (1-\kappa_3)(\kappa_5-\kappa_3) + \bar{c}_2 \kappa_5 (\kappa_3-\kappa_1)(\kappa_4-\kappa_3)}{\kappa_3 \kappa_5 (1-\kappa_3)(\kappa_5-\kappa_3)} \quad (\text{A.11b})$$

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